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# FUNDAMENTAL SOLUTION IN MICROPOLAR VISCOTHERMOELASTIC SOLIDS WITH VOID

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In the present article, we construct the fundamental solution to a system of differential equations in micropolar viscothermoelastic solids with voids in case of steady oscillations in terms of elementary functions. Some basic properties of the fundamental solution are also established.

**Key words:** micropolar thermodiffusion, steady oscillations, visothemoelastic, micropolar with void.

## 1. Introduction

The theory of elasticity concerning elastic materials consisting of vacous pores (voids) distributed throughout the body has become very important due to its theoretical and practical relevance. Problems concerning voids play a vital role in the practical problems of geological and synthetic porous media where the classical theory is inadequate. Mackenzie (1950) was perhaps the first to estimate the effective elastic moduli of a linearly elastic isotropic materials with voids. Cowin and Nunziato (1983) established the linear theory of elasticity with voids, while Nunziato and Cowin (1979) derived the non-linear theory of elastic materials with voids. Puri and Cowin (1985) studied the behavior of plane harmonic waves in linear elastic materials with voids. Chandrasekharaiah (1987a; b) investigated the effect of voids on Rayleigh-Lamb waves in a homogeneous elastic plate with voids. Results on linear and non-linear problems in thermoelastic with voids have been obtained by many researchers such as Dhaliwal and Wang (1994); Scarpetta (1995), Iesan and Quintanilla (1995); Quintanilla (2001). Wright (1998) studied the relationship between theories of effective moduli and dynamical theory of materials with voids. Singh and Tomar (2007) studied the propagation of plane waves in an infinite thermoelastic medium with voids using the theory developed by Iesan (1986).

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The basic assumption in classical continuum mechanics is that the effect of microstructure of a material is not essential for describing the mechanical behavior. Such an approximation has been shown in many well known cases. However, discrepancies between the classical theory and experimental are often observed indicating that microstructure might be important. A theory in which some considerations are given to microstructure is the theory of micropolar continuum mechanics. The difference between the micropolar and classical theory of elasticity is related to the fact that the deformation and microrotation have six degrees of fredom. Eringen (1968; 1986) gave a complete description of the linear theory of micropolar elasticity . for application ,it can be modeled composites with rigid choped fibers, elastic solids with granular inclusion and industrial materials such as liquid crystals. The linear theory of micropolar thermoelasticity was developed by extending the theory of micropolar continua to include the thermal effect. A comprehensive review of the subject was given by Eringen (1970; 1999) and Nowacki (1986).

Ciarletta and Straughan (2007) presented a model for coupled elasto-acoustic waves, thermal waves and waves associated with voids, in a porous medium. Miglani and Kaushal (2011) investigated two dimensional problems in a micropolar elastic medium with voids. Kumar and Panchal (2011) studied circular crested waves in a micropolar porous medium possessing cubic symmetry. Kumar *et al*. (2012) investigated deformations due to various sources in a micropolar elastic solid with voids under inviscid liquid half space. Aouadi (2012a) studied the uniqueness and existence theorems in thermoelaticity with voids without energy dissipation. Aouadi *et al.* (2012b) studied the problem of exponential decay in thermoelastic materials with voids with dissipative boundary without thermal dissipation.

The inelastic behavior of the earth's material plays an important role in changing the characteristics of seismic waves. The general theory of viscoelasticity describes the linear behavior of both elastic and inelastic materials and provides the basis for describing the attenuation of seismic waves due to inelasticity.

Eringen (1967) extended the theory of micropolar elasticity to obtain the linear constitutive theory for a micropolar material possessing internal friction. The problem of micropolar viscoelastic waves was discussed by McCarthy and Eringen (1969). They discussed the propagation conditions and growth equations governing the propagation of waves in a micropolar viscoelastic medium. Manole (1988) established the uniqueness theorem in the theory of linear viscoelasticity and in the theory of micropolar linear visoelasticity by using the Laplace transform technique. Manole (1992) presented variational theorems in the linear micropolar viscoelastic solid.

Gale (2000) studied Saint-Venant's problem of micropolar viscoelasticity. Kumar (2000) investigated wave propagation in a micropolar viscoelastic generalized thermoelastic solid. Dynamical problems of micropolar visoelasticity was discussed by Kumar and Chaudhary (2001). Kumar and Chaudhary (2005a; b) studied the deformation and disturbance due to a time harmonic source in an orthotropic micropolar viscoelastic medium. Kumar and Partap (2010) investigated Rayliegh-Lamb waves in microstrech viscoelastic media. Ezzat and Atef (2011) investigated the problem of a magnetothermo-viscoelastic material with a spherical cavity. Svanadze (2012) studied the problem of a potential method in the linear theories of viscoelasticity and thermoviscoelasticity for Kelvin-Voigt materials. Luppe *et al.* (2012) investigated the problem of effective wave numbers for thermoviscoelastic media containing a random configuration of spherical scatters.

To investigate the boundary value problems of the theory of elasticity and thermoelasticity by the potential method, it is necessary to construct the fundamental solution to a system of partial differential equations and to establish their basic properties respectively. Hetnarski (1964a; b) was the first to study the fundamental solutions in the classical theory of coupled thermoelasticity. The fundamental solutions in the microcontinuum field theories were constructed by Svanadze (1988; 1996; 2004a; b; 2007). The information related to fundamental solutions of differential equations is contained in the books of Hőrmander (1963; 1983). Kumar and Kansal (2012a) discussed the plane wave and the fundamental solution in the generalized theories of thermoelastic diffusion.

In this paper, the fundamental solution to a system of differential equations in the case of steady oscillation in terms of elementary functions has been considered.

## 2. Basic equations

Following Eringen (1967), Iesan (2011) and Lord-Shulman (1967), the basic equations in a homogeneous isotropic micropolar viscothermoelastic solid in the absence of body force, body couple, equilibrated force and heat sources are

$$(\mu_0+K_0)\Delta \boldsymbol{u}+(\mu_0+\lambda_0)\nabla\nabla\cdot\boldsymbol{u}+K_0(\nabla\times\boldsymbol{\varphi})+b_0\nabla\Phi-\beta_1\nabla T=\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}, \tag{2.1}$$

$$(\gamma_0\Delta-2K_0)\boldsymbol{\varphi}+(\alpha_0+\beta_0)\nabla(\nabla\cdot\boldsymbol{\varphi})+K_0(\nabla\times\boldsymbol{u})=\rho j\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2}, \tag{2.2}$$

$$\alpha_0^*\Delta\Phi-\gamma_0^*\nabla\cdot\boldsymbol{u}-\xi_0\Phi+\left(\tau^*\Delta+m\right)T=\rho\chi\frac{\partial^2\Phi}{\partial t^2}, \tag{2.3}$$

$$K_1\Delta T-\beta_1 T_0\left(1+\tau_0\frac{\partial}{\partial t}\right)\nabla\cdot\dot{\boldsymbol{u}}-\rho C^*\left(1+\tau_0\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial t}+\left[\varsigma\Delta-mT_0\left(1+\tau_0\frac{\partial}{\partial t}\right)\right]\frac{\partial\Phi}{\partial t}=0 \tag{2.4}$$

where

$$\lambda_0=\lambda+\lambda^*\frac{\partial}{\partial t},\quad \mu_0=\mu+\mu^*\frac{\partial}{\partial t},\quad K_0=K+K^*\frac{\partial}{\partial t},\quad \alpha_0=\alpha+\alpha^*\frac{\partial}{\partial t},$$
$$\beta_0=\beta+\beta^*\frac{\partial}{\partial t},\quad \gamma_0=\gamma+\gamma^*\frac{\partial}{\partial t},\quad \xi_0=\xi+\xi^*\frac{\partial}{\partial t}. \tag{2.5}$$

In these relations $\rho$ is the density, $\boldsymbol{u}$ is the displacement vector, $\boldsymbol{\varphi}$ is the microrotation vector, $\Phi$ is the volume fraction field, $j$ is microinertia, $C$ is the specific heat at constant strain, $T=\Theta-T_0$ is a small temperature increment, $\Theta$ is the absolute temperature of the medium, $T_0$ is the reference temperature of the body choosen such that $\left|\frac{T}{T_0}\right|<<1$, $\beta_1=(3\lambda_0+2\mu_0+K_0)(\alpha_t)$, where $\alpha_t$ is the coefficient of linear thermal expension, $\Delta=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}$, $\nabla=\left(\hat{i}\frac{\partial}{\partial x_1}+\hat{j}\frac{\partial}{\partial x_2}+\hat{k}\frac{\partial}{\partial x_3}\right)$, $\tau_0$ is the relaxation time, $\chi$ is the equilibrated inertia, $K_1$ is thermal conductivity and $\lambda,\mu,K,\alpha,\beta,\gamma,\xi,\varsigma,\lambda^*,\mu^*,K^*,\alpha^*,\beta^*,\gamma^*,\xi^*,\tau^*$ are material constants of the theory, $\tau_0=0$ for the theory of coupled viscothermoelasticity model.

Let $X=(x_1,x_2,x_3)$ be the point of the Euclidean three-dimensional space $E^3$, $|x|=\left(x_1^2,x_2^2,x_3^2\right)^{\frac{1}{2}}$, $D_x=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)$.

We define the following dimensionless quantities as

$$x'=\frac{\omega_1^*}{c_1}x,\quad \boldsymbol{u}'=\frac{\rho\omega_1^* c_1}{\beta_1 T_0}\boldsymbol{u},\quad \boldsymbol{\varphi}'=\frac{\rho c_1^2}{\beta_1 T_0}\boldsymbol{\varphi},$$
$$T'=\frac{T}{T_0},\quad t'=\omega_1^* t,\quad \Phi'=\frac{\chi\rho\omega_1^{*2}c_1}{\beta_1 T_0}\Phi \tag{2.6}$$

where

$$\omega_1^* = \frac{\rho C^* c_1^2}{K_1}, \qquad c_1^2 = \frac{\lambda + 2\mu + K}{\rho},$$

and $\omega_1^*$ is the characteristic frequency.

Making use of these dimensionless quantities defined by Eq.(2.6) in Eqs (2.1)-(2.4), after suppressing the primes yields

$$\delta_1 \Delta \boldsymbol{u} + \delta_2 \nabla\nabla \cdot \boldsymbol{u} + \delta_3 \left(\nabla \times \boldsymbol{\varphi}\right) + \delta_4 \nabla\Phi - \nabla T = \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2}, \tag{2.7}$$

$$\left(\delta_5 \Delta - 2\delta_3\right)\boldsymbol{\varphi} + \delta_6 \nabla\left(\nabla \cdot \boldsymbol{\varphi}\right) + \delta_3 \left(\nabla \times \boldsymbol{u}\right) = \delta_7 \frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2}, \tag{2.8}$$

$$\left(\delta_8 \Delta - \delta_9\right)\Phi - \delta_{10} \nabla \cdot \boldsymbol{u} + \left(\delta_{11}\Delta + \delta_{12}\right)T = \frac{\partial^2 \Phi}{\partial t^2}, \tag{2.9}$$

$$\left(\Delta - \tau_t^0\right)\frac{\partial T}{\partial t} - \tau_t^0 \varsigma_2 \nabla \cdot \boldsymbol{u} - \left(\varsigma_3 \Delta - \varsigma_4 \tau_t^0\right)\frac{\partial \Phi}{\partial t} = 0\,. \tag{2.10}$$

We assume the displacement vector, microrotation vector, volume fraction field and temperature change as

$$\left[\boldsymbol{u}(x,t),\ \boldsymbol{\varphi}(x,t),\ \Phi(x,t),\ T(x,t)\right] = \mathrm{Re}[\bar{\boldsymbol{u}}, \bar{\varphi}, \bar{\Phi}, \bar{T}]e^{-i\omega t} \tag{2.11}$$

where $\omega$ is the frequency.

Making use of Eq.(2.11) in Eqs (2.7)-(2.10) and after ommiting the bars, we obtain the system of equations for steady oscillation as

$$\left(\delta_1 \Delta + \omega^2\right)\boldsymbol{u} + \delta_2 \nabla\left(\nabla \cdot \boldsymbol{u}\right) + \delta_3 \left(\nabla \times \boldsymbol{\varphi}\right) + \delta_4 \nabla\Phi - \nabla T = 0, \tag{2.12}$$

$$\left(\delta_5 \Delta + \mu^*\right)\boldsymbol{\varphi} + \delta_6 \nabla\left(\nabla \cdot \boldsymbol{\varphi}\right) + \delta_3 \left(\nabla \times \boldsymbol{u}\right) = 0, \tag{2.13}$$

$$\left(\delta_8 \Delta + \mu_1^*\right)\Phi - \delta_{10} \nabla\left(\nabla \cdot \boldsymbol{u}\right) + \left(\delta_{11}\Delta + \delta_{12}\right)T = 0, \tag{2.14}$$

$$\left(\Delta - \tau_t^{\ 10}\right)T - \varsigma_2 \tau_t^{10}\left(\nabla \cdot \boldsymbol{u}\right) + \left[\varsigma_3(-i\omega)\Delta - \varsigma_4 \tau_t^{\ 10}\right]\Phi = 0 \tag{2.15}$$

where

$$\mu^* = \delta_7 \omega^2 - 2\delta_3, \qquad \mu_1^* = \omega^2 - \delta_9, \qquad \tau_t^{\ 10} = -i\omega\left(1 - \tau_0 i\omega\right),$$

$$\delta_1=\frac{(\mu_0+K_0)}{(\lambda_0+2\mu_0+K_0)},\quad \delta_2=\frac{(\mu_0+\lambda_0)}{(\lambda_0+2\mu_0+K_0)},\quad \delta_3=\frac{K_0}{(\lambda_0+2\mu_0+K_0)},\quad \delta_4=\frac{b_0}{\rho\chi\omega_1^{*2}},$$

$$\delta_5=\frac{\gamma_0\omega_1^*}{\rho c_1^4},\quad \delta_6=\frac{(\alpha_0+\beta_0)\omega_1^{*2}}{\rho c_1^4},\quad \delta_7=\frac{j\omega_1^{*2}}{c_1^2},\quad \delta_8=\frac{\alpha_0^*}{\chi\rho c_1^2},\quad \delta_9=\frac{\xi_0}{\chi\rho\omega_1^{*2}},\quad \delta_{10}=\frac{\gamma_0^*}{\rho c_1^2},$$

$$\delta_{11}=\frac{\tau^*\omega_1^{*2}}{\beta_1 c_1^2},\qquad \delta_{12}=\frac{m}{\beta_1}.$$

Introducing the matrix differential operator

$$\boldsymbol{F}(D_x)=\left\|F_{gh}(D_x)\right\|_{8\times 8} \tag{2.16}$$

where

$$F_{mn}(D_x)=\left[\delta_1\Delta+\omega^2\right]\delta_{mn}+\delta_2\frac{\partial^2}{\partial x_m\partial x_n},\qquad F_{m,n+3}(D_x)=F_{m+3,n}(D_x)=\delta_3\sum_{r=1}^{3}\varepsilon_{mrn}\frac{\partial}{\partial r},$$

$$F_{m7}(D_x)=\delta_4\frac{\partial}{\partial x_n},\qquad F_{m8}(D_x)=-\frac{\partial}{\partial x_m},$$

$$F_{m+3,n+3}(D_x)=\left[\delta_5\Delta+\mu^*\right]\delta_{mx}+\delta_6\frac{\partial^2}{\partial x_m\partial x_n},\qquad F_{m+3,7}(D_x)=F_{7,n+3}=F_{m+3,8}=F_{8,n+3}=0,$$

$$F_{7n}(D_x)=\left(\delta_8\Delta+\mu_1^*\right),\qquad F_{77}=-\delta_{10}\frac{\partial^2}{\partial x_m\partial x_n},\qquad F_{78}=\left(\delta_{11}\Delta+\delta_{12}\right),$$

$$F_{8n}=\left(\Delta-\tau_t^{10}\right),\qquad F_{87}(D_x)=-\varsigma_2\tau_t^{10}\frac{\partial}{\partial x_n},\qquad F_{88}=\left[(-i\omega)\varsigma_3\Delta-\varsigma_4\tau_t^{10}\right].$$

Here $\varepsilon_{mrn}$ is the alternating tensor and $\delta_{mn}$ is the Kronecker delta function.
The system of Eqs (2.11)-(2.14) can be written as

$$F(D_x)U(x)=0 \tag{2.18}$$

where $U=(\boldsymbol{u},\ \boldsymbol{\varphi},\ \Phi,\ T)$ is an eight-component vector function on $E^3$.
Assuming that

$$\delta_1\delta_5\delta_8\neq 0\,. \tag{2.19}$$

If condition (2.19) is satisfied, then $F$ is an elliptic differential operator given by Hormader (1963). The fundamental solution to the system of Eqs (2.12)-(2.15), (The fundamental matrix of operator $F$) is the matrix $G(x)=\left\|G_{gh}(x)\right\|_{8\times 8}$ satisfying conditions

$$F(D_x)G(x)=\delta(x)I(x) \tag{2.20}$$

where $\delta$ is the Dirac delta function, $I=\left\|\delta_{gh}\right\|_{8\times 8}$ is the unit matrix and $x\in E^3$.

## 2.1. Fundamental solution to the system of equations of steady oscillations

We consider the system of equations

$$\delta_1\Delta\boldsymbol{u}+\delta_2\nabla(\nabla\cdot\boldsymbol{u})+\delta_3(\nabla\times\boldsymbol{\varphi})+\delta_{10}\nabla\Phi-\varsigma_2\tau_t{}^{10}\nabla T+\omega^2\boldsymbol{u}=H', \tag{2.21}$$

$$(\delta_5\Delta+\mu^*)\boldsymbol{\varphi}+\delta_6\nabla(\nabla\cdot\boldsymbol{\varphi})+\delta_3(\nabla\times\boldsymbol{u})=H'', \tag{2.22}$$

$$\left(\delta_8\Delta+\mu_1{}^*\right)\Phi-\delta_4\nabla(\nabla\cdot\boldsymbol{u})+\left(i\omega\varsigma_3\delta_{11}\Delta+\varsigma_4\tau_t{}^{10}\right)T=L, \tag{2.23}$$

$$\left(\Delta-\tau_t{}^{10}\right)T-(\nabla\cdot\boldsymbol{u})+\left[\delta_{11}\Delta+\delta_{12}\right]\Phi=M \tag{2.24}$$

where $H'$ and $H''$ are three component vector functions on $E^3$.

The system of Eqs (2.21)-(2.24) may be written in the form

$$F^{tr}(D_x)U(x)=Q(x) \tag{2.25}$$

where $F^{tr}$ is the transpose of the matrix $F, Q=(H',H'',L,M)$ and $x\in E^3$.

Applying the operator div. to Eqs (2.21) and (2.22), we obtain

$$\left(\Delta+\omega^2\right)\nabla\cdot\boldsymbol{u}+\delta_{10}\Delta\Phi-\varsigma_2\tau_t{}^{10}\Delta T=\nabla\cdot H', \tag{2.26}$$

$$\left(\delta^*\Delta+\mu^*\right)\nabla\cdot\boldsymbol{\varphi}=\nabla\cdot H'', \tag{2.27}$$

$$\left(\delta_8\Delta+\mu_1{}^*\right)\Phi-\delta_4(\nabla\cdot\boldsymbol{u})-\left(i\omega\varsigma_3\delta\Delta+\varsigma_4\tau_t{}^{10}\right)T=L, \tag{2.28}$$

$$\left(\Delta-\tau_t{}^{10}\right)T-(\nabla\cdot\boldsymbol{u})+\left[\delta_{11}\Delta+\delta_{12}\right]\Phi=M \tag{2.29}$$

where

$$\delta^*=\delta_5+\delta_6\,.$$

Equations (2.26), (2.28), (2.29) may be written in the form

$$\Gamma_1(\Delta)\nabla\cdot\boldsymbol{u}=\Psi_1,\qquad \Gamma_1(\Delta)\Phi=\Psi_2,\qquad \Gamma_1(\Delta)T=\Psi_3 \tag{2.30}$$

where

$$\Gamma_1(\Delta)=\begin{vmatrix}\left(\Delta+\omega^2\right) & \delta_{10}\Delta & -\varsigma_2\tau_t^{\ 10}\Delta\\ -\delta_4 & \left(\delta_8\Delta+\mu_1^{\ *}\right) & -\left(i\omega\varsigma_3\Delta+\varsigma_4\tau_t^{\ 10}\right)\\ -1 & \left(\delta_{11}\Delta+\delta_{12}\right) & \left(\Delta-\tau_t^{\ 10}\right)\end{vmatrix},$$

$$\begin{gathered}\Psi_1=\left[\left(\delta_8\Delta+\mu_1^{\ *}\right)\left(\Delta-\tau_t^{\ 10}\right)+\left(i\omega\varsigma_3\Delta+\varsigma_4\tau_t^{\ 10}\right)\left(\delta_{11}\Delta+\delta_{12}\right)\right]\nabla\cdot H'+\\ -\Delta L\left[\delta_{10}\left(\Delta-\tau_t^{\ 10}\right)+\varsigma_2\tau_t^{\ 10}\left(\delta_{11}\Delta+\delta_{12}\right)\right]-\Delta M\left[\delta_{10}\left(i\omega\varsigma_3\Delta+\varsigma_4\tau_t^{\ 10}\right)-\varsigma_2\tau_t^{\ 10}\left(\delta_8\Delta+\mu_1^{\ *}\right)\right],\end{gathered} \tag{2.31}$$

$$\begin{gathered}\Psi_2=\left[\delta_4\left(\Delta-\tau_t^{\ 10}\right)+i\omega\varsigma_3+\varsigma_4\tau_t^{\ 10}\right]\nabla\cdot H'+\left[\left(\Delta+\omega^2\right)\left(\Delta-\tau_t^{\ 10}\right)-\varsigma_2\tau_t^{\ 10}\Delta\right]L+\\ +\left[\left(\Delta+\omega^2\right)\left(i\omega\varsigma_3\Delta+\varsigma_4\tau_t^{\ 10}\right)+\varsigma_2\tau_t^{\ 10}\delta_4\right]M,\end{gathered} \tag{2.32}$$

$$\begin{gathered}\Psi_3=\left[-\delta_4\left(\delta_{11}\Delta+\delta_{12}\right)+\left(\delta_8\Delta+\mu_1^{\ *}\right)\right]\nabla\cdot H'-\left[\left(\Delta+\omega^2\right)\left(\delta_{11}\Delta+\delta_{12}\right)+\delta_{10}\Delta\right]L+\\ +\left[\left(\Delta+\omega^2\right)\left(\delta_8\Delta+\mu_1^{\ *}\right)+\delta_4\delta_{10}\Delta\right]M.\end{gathered} \tag{2.33}$$

It is noticed that

$$\Gamma_1(\Delta)=\prod_{m=1}^{m}\left(\Delta+\lambda_m^2\right) \tag{2.34}$$

where
$\lambda_m^2$, $m=1, 2, 3$ are the roots of the equation $\Gamma(-\kappa)=0$ (with respect to $\kappa$)
From Eq.(2.27), it follows that

$$\left(\Delta+\lambda_6^2\right)\nabla\cdot\boldsymbol{\varphi}=\frac{1}{\delta^*}\nabla\cdot H'' \tag{2.35}$$

where $\lambda_6^2=\dfrac{\mu^*}{\delta^*}$.

Applying the operator $\left(\delta_5\Delta+\mu^*\right)$ and $\delta_3\mathrm{curl}$ on Eqs (2.21) and (2.22), we obtain

$$\begin{gathered}\left(\delta_5\Delta+\mu^*\right)\left[\delta_1\Delta\boldsymbol{u}+\delta_2\nabla\nabla\cdot\boldsymbol{u}+\omega^2\boldsymbol{u}\right]+\delta_3\left(\delta_5\Delta+\mu^*\right)\nabla\times\boldsymbol{\varphi}=\\ =\left(\delta_5\Delta+\mu^*\right)\left[H'-\delta_{10}\nabla\Phi+\varsigma_2\tau_t^{\ 10}\nabla T\right],\end{gathered} \tag{2.36}$$

$$\delta_3\left(\delta_5\Delta+\mu^*\right)\nabla\times\boldsymbol{\varphi}=-\delta_3^2\nabla\times\nabla\times\boldsymbol{u}+\delta_3\nabla\times H'', \tag{2.37}$$

$$\delta_3\left(\delta_5\Delta+\mu^*\right)\nabla\times\boldsymbol{\varphi}=-\delta_3^2\left[\nabla\nabla\cdot\boldsymbol{u}-\Delta u\right]+\delta_3\nabla\times H''. \tag{2.38}$$

Making use of Eq.(2.40) in Eq.(2.38), yields

$$\begin{aligned}&\left(\delta_5\Delta+\mu^*\right)\left[\delta_1\Delta\boldsymbol{u}+\delta_2\nabla\nabla\cdot\boldsymbol{u}+\omega^2\boldsymbol{u}\right]-\delta_3^2\left[\nabla\nabla\cdot\boldsymbol{u}-\Delta\boldsymbol{u}\right]=\\&=\left(\delta_5\Delta+\mu^*\right)\left[H'-\delta_{10}\nabla\Phi+\varsigma_2\tau_t{}^{10}\nabla T\right]-\delta_3\nabla\times H''.\end{aligned} \tag{2.39}$$

Equation (2.39) can be written as

$$\begin{aligned}&\left[\left\{\left(\delta_5\Delta+\mu^*\right)\delta_1+\delta_3^2\right\}\Delta+\left(\delta_5\Delta+\mu^*\right)\omega^2\right]\boldsymbol{u}=\\&=-\left[\delta_2\left(\delta_5\Delta+\mu^*\right)-\delta_3^2\right]\nabla\nabla\cdot\boldsymbol{u}+\left(\delta_5\Delta+\mu^*\right)\left[H'-\delta_{10}\nabla\Phi+\varsigma_2\tau_t{}^{10}\nabla T\right]-\delta_3\nabla\times H''.\end{aligned} \tag{2.40}$$

Applying the operator $\Gamma_1(\Delta)$ to Eq.(2.40) and using Eq.(2.30), we obtain

$$\begin{aligned}&\Gamma_1(\Delta)\left[\left\{\delta_1\delta_5\Delta+\left(\mu^*\delta_1+\delta_3^2+\delta_5\omega^2\right)\Delta+\mu^*\omega^2\right\}\right]\boldsymbol{u}=\\&=-\left[\delta_2\left(\delta_5\Delta+\mu^*\right)-\delta_3^2\right]\nabla\Psi_1+\left(\delta_5\Delta+\mu^*\right)\left[\Gamma_1(\Delta)H'-\delta_{10}\nabla\Psi_2+\varsigma_2\tau_t{}^{10}\nabla\Psi_3\right]+\\&-\delta_3\Gamma_1(\Delta)\nabla\times H''.\end{aligned} \tag{2.41}$$

The above equation may be written in the form

$$\Gamma_1(\Delta)\Gamma_2(\Delta)=\Psi' \tag{2.42}$$

where

$$\begin{aligned}&\Gamma_1(\Delta)=f^*\begin{Vmatrix}\delta_1\Delta+\omega^2 & \delta_3\Delta\\ -\delta_3 & \delta_5\Delta+\mu^*\end{Vmatrix}_{2\times 2}, \qquad f^*=\frac{1}{\delta_1\delta_5},\\&\Psi'=f^*\begin{bmatrix}\left[-\delta_2\left(\delta_5\Delta+\mu^*\right)+\delta_3^2\Delta\right]\nabla\Psi_1+\\+\left(\delta_5\Delta+\mu^*\right)\left[\Gamma_1(\Delta)H'-\delta_{10}\nabla\Psi_2+\varsigma_2\tau_t{}^{10}\nabla\Psi_3\right]-\delta_3\Gamma_1(\Delta)\nabla\times H''\end{bmatrix}.\end{aligned} \tag{2.43}$$

It is noticed that

$$\Gamma_2(\Delta)=\left(\Delta+\lambda_4^2\right)\left(\Delta+\lambda_5^2\right) \tag{2.44}$$

where $\lambda_4^2, \lambda_5^2$ are the roots of the $\Gamma_2(-\kappa)$ w.r.t $\kappa$.

Applying the operator $\delta_3 \text{curl}$ to Eq.(2.21) and $\delta_1\Delta + \omega^2$ to Eq.(2.22) respectively , we obtain

$$\delta_3\left(\delta_1\Delta + \omega^2\right)\nabla\times\boldsymbol{u} + \delta_3^2\nabla\times\nabla\times\boldsymbol{\varphi} = \delta_3\nabla\times H',$$

i.e.,

$$\delta_3\left(\delta_1\Delta + \omega^2\right)\nabla\times\boldsymbol{u} = \delta_3\nabla\times H' - \delta_3^2\nabla\times\nabla\times\boldsymbol{\varphi}, \tag{2.45}$$

$$\left(\delta_1\Delta + \omega^2\right)\left(\delta_5\Delta + \mu^*\right)\boldsymbol{\varphi} + \left(\delta_1\Delta + \omega^2\right)\delta_6\nabla\nabla\cdot\boldsymbol{\varphi} + \left(\delta_1\Delta + \omega^2\right)\delta_3\left(\nabla\times\boldsymbol{u}\right) = \left(\delta_1\Delta + \omega^2\right)H'', \tag{2.46}$$

we know the identity

$$\nabla\times\nabla\times\boldsymbol{\varphi} = \nabla\nabla\cdot\boldsymbol{\varphi} - \Delta\boldsymbol{\varphi}. \tag{2.47}$$

Making use of Eqs (2.45) and (2.47) in Eq.(2.46), yields

$$\begin{aligned}&\left(\delta_1\Delta + \omega^2\right)\left(\delta_5\Delta + \mu^*\right)\boldsymbol{\varphi} + \left(\delta_1\Delta + \omega^2\right)\delta_6\nabla\nabla\cdot\boldsymbol{\varphi} + \delta_3^2\Delta - \delta_3^2\nabla\nabla\cdot\boldsymbol{\varphi} = \\ &= \left(\delta_1\Delta + \omega^2\right)H'' - \delta_3\nabla\times H'.\end{aligned} \tag{2.48}$$

Equation (2.48) can be written as

$$\begin{aligned}&\left[\left\{\delta_1\left(\delta_5\Delta + \mu^*\right) + \delta_3^2\right\}\Delta + \left(\delta_5\Delta + \mu^*\right)\omega^2\right]\boldsymbol{\varphi} = \\ &= -\left[\left(\delta_1\Delta + \omega^2\right)\delta_6 - \delta_3^2\right]\nabla\nabla\cdot\boldsymbol{\varphi} + \left(\delta_1\Delta + \omega^2\right)H'' - \delta_3\nabla\times H'.\end{aligned} \tag{2.49}$$

Applying the operator $\left(\Delta + \lambda_6^2\right)$ to Eq.(2.49) and using Eq.(2.35), we obtain

$$\begin{aligned}&\left(\Delta + \lambda_6^2\right)\left[\delta_1\delta_5\Delta^2 + \Delta\left(\delta_3^2 + \omega^2\delta_5 + \delta_1\mu^*\right) + \mu^*\omega^2\right]\boldsymbol{\varphi} = \\ &= -\frac{1}{\delta^*}\left[\left(\delta_1\Delta + \omega^2\right)\delta_6 - \delta_3^2\right]\nabla\nabla\cdot H'' + \left(\delta_1\Delta + \omega^2\right)\left(\Delta + \lambda_6^2\right)H'' - \delta_3\left(\Delta + \lambda_6^2\right)\nabla\times H'.\end{aligned} \tag{2.50}$$

The above equation can be written as

$$\Gamma_2(\Delta)\left(\Delta + \lambda_6^2\right)\boldsymbol{\varphi} = \Psi'' \tag{2.51}$$

where

$$\begin{aligned}&\Psi'' = f^*\left[-\delta_3\left(\Delta + \lambda_6^2\right)\nabla\times H' + \left(\delta_1\Delta + \omega^2\right)\left(\Delta + \lambda_6^2\right)H'' + \right. \\ &\left. -\frac{1}{\delta^*}\left[\left(\delta_1\Delta + \omega^2\right)\delta_6 - \delta_3^2\right]\nabla\nabla\cdot H''.\right.\end{aligned} \tag{2.52}$$

From Eqs (2.30), (2.42) and (2.51) results

$$\Theta(\Delta)\bigcup(x)=\hat{\Psi}(x) \tag{2.53}$$

where $\hat{\Psi}=\left(\Psi',\Psi'',\Psi_3,\Psi_4\right)$ and

$$\Theta=\left\|\Theta_{gh}(\Delta)\right\|_{8\times 8},$$

$$\Theta_{mn}(\Delta)=\Gamma_1(\Delta)\Gamma_2(\Delta)=\prod_{q=1}^{5}\left(\Delta+\lambda_q^2\right),$$

$$\Theta_{m+3,n+3}(\Delta)=\Gamma_2\left(\Delta+\lambda_6^2\right)=\prod_{q=4}^{6}\left(\Delta+\lambda_q^2\right),$$

$$\Theta_{gh}=0, \qquad \Theta_{77}(\Delta)=\Theta_{88}(\Delta)=\Gamma_1(\Delta).$$

The above Eqs (2.32), (2.33), (2.43) and (2.52) can be rewritten in the matrix form

$$\Psi'=\left[f^*\left(\delta_5\Delta+\mu^*\right)\Gamma_1(\Delta)J+q_{11}(\Delta)\nabla\nabla\right]H+q_{21}(\Delta)\nabla\times H''+q_{31}(\Delta)\nabla L+q_{41}(\Delta)\nabla M, \tag{2.54}$$

$$\Psi''=q_{12}(\Delta)\nabla\times H'+\left[f^*\left(\Delta+\lambda_6^2\right)\left(\delta_1\Delta+\omega^2\right)J+q_{22}(\Delta)\nabla\nabla\right]H'', \tag{2.55}$$

$$\Psi_2=q_{13}\nabla\cdot H'+q_{33}(\Delta)L+q_{43}(\Delta)M, \tag{2.56}$$

$$\Psi_3=q_{14}\nabla\cdot H'+q_{34}(\Delta)L+q_{44}(\Delta)M \tag{2.57}$$

where

$$q_{12}(\Delta)=-f^*\delta_3^*\left(\Delta+\lambda_6^2\right), \qquad q_{22}(\Delta)=-\frac{f^*}{\delta^*}\left[\delta_6\left(\delta_1\Delta+\omega^2\right)-\delta_3^2\right],$$

$$q_{13}=\delta_4\left(\Delta-\tau_t^{10}\right)+\left(i\omega\varsigma_3\Delta+\varsigma_4\tau_t^{10}\right), \qquad q_{33}=\left(\Delta+\omega^2\right)\left(\Delta-\tau_t^{10}\right)-\varsigma_t\tau_t^{10}\Delta,$$

$$q_{43}=\left(\Delta+\omega^2\right)\left(i\omega\varsigma_3\Delta+\varsigma_4\tau_t^{10}\right)+\varsigma_2\tau_t^{10}\delta_4, \qquad q_{14}=-\delta_4\left(\delta_{11}\Delta+\delta_{12}\right)+\left(\delta_8\Delta+\mu_1^*\right),$$

$$q_{34}=-\left(\Delta+\omega^2\right)\left(\delta_{11}\Delta+\delta_{12}\right)+\delta_{10}\Delta, \qquad q_{44}=-\left(\Delta+\omega^2\right)\left(\delta_8\Delta+\mu_1^*\right)+\delta_4\delta_{10}\Delta,$$

$$q_{11}(\Delta)=f^*\left(\delta_5\Delta+\mu^*\right)\left[-\delta_2\left(R_5R_8+R_6R_7\right)+\delta_{10}\left(R_4R_8+R_6\right)+\varsigma_2\tau_t^{10}\left(-R_4R_7+R_5\right)\right]+$$
$$+\delta_3^2\Delta\left(R_5R_8+R_6R_3\right),$$

$$q_{31}(\Delta)=f^*\left(\delta_5\Delta+\mu^*\right)\left[\delta_2\left(R_2R_8+R_3R_7\right)+\delta_{10}\left(R_1R_6-R_3\nabla\right)-\varsigma_2\tau_t^{10}\left(R_1R_7+R_2\nabla\right)\right]+$$
$$-f^*\delta_3^2\Delta\left(R_2R_8+R_3R_7\right),$$

$$q_{41}(\Delta)=f^*\left(\delta_5\Delta+\mu^*\right)\left[-\delta_2\left(-R_2R_6+R_3R_8\right)-\delta_{10}\left(R_1R_6+R_3R_4\nabla\right)+\varsigma_2\tau_t^{10}\left(R_1R_5+R_2R_4\nabla\right)\right]$$
$$+f^*\delta_3^2\Delta\left(-R_2R_6+R_3R_8\right),$$

$$\nabla\nabla\cdot H'=f^*\left[-\delta_2\left(\delta_5\Delta+\mu^*\right)+\delta_3^2\Delta\right]\left(R_5R_8+R_6R_7\right)-f^*\left(\delta_5\Delta+\mu^*\right)\delta_{10}\left(R_4R_8+R_6\right)+$$
$$+f^*\left(\delta_5\Delta+\mu^*\right)\varsigma_2\tau_t^{10}\left(-R_4R_7+R_5\right),$$

$$\nabla L=f^*\left[-\delta_2\left(\delta_5\Delta+\mu^*\right)+\delta_3^2\Delta\right]\left(R_2R_8+R_3R_7\right)+f^*\left(\delta_5\Delta+\mu^*\right)\delta_{10}\left(R_1R_6-R_3\nabla\right)+$$
$$-f^*\left(\delta_5\Delta+\mu^*\right)\varsigma_2\tau_t^{10}\left(R_1R_7+R_2\nabla\right),$$

$$\nabla M=f^*\left[-\delta_2\left(\delta_5\Delta+\mu^*\right)+\delta_3^2\Delta\right]\left(-R_2R_6+R_3R_8\right)-f^*\left(\delta_5\Delta+\mu^*\right)\delta_{10}\left(R_1R_6+R_3R_4\nabla\right)+$$
$$+f^*\left(\delta_5\Delta+\mu^*\right)\varsigma_2\tau_t^{10}\left(R_1R_5+R_2R_4\nabla\right).$$

From Eqs (2.54)-(2.57), we have

$$\Psi(x)=R^{tr}(D_x)Q(x) \tag{2.58}$$

where

$$R_{mn}\left(D_x\right)=f^*\left[\delta_5\Delta+\mu^*\right]\Gamma_1(\Delta)\delta_{mn}+q_{11}(\Delta)\frac{\partial^2}{\partial x_m\partial x_n},\qquad R_{m,n+3}\left(D_x\right)=q_{12}(\Delta)\sum_{r=1}^{3}\varepsilon_{mrn}\frac{\partial}{\partial x_r},$$

$$R_{m7}\left(D_x\right)=q_{13}(\Delta)\frac{\partial}{\partial x_m},\qquad R_{m8}\left(D_x\right)=q_{14}(\Delta)\frac{\partial}{\partial x_m},\qquad R_{m+3,n}\left(D_x\right)=q_{21}(\Delta)\sum_{r=1}^{3}\varepsilon_{mrn}\frac{\partial}{\partial x_r},$$

$$R_{m+3,n+3}\left(D_x\right)=f^*\left(\Delta+\lambda_6^2\right)\left(\delta_1\Delta+\omega^2\right)\delta_{mn}+q_{22}(\Delta)\frac{\partial^2}{\partial x_m\partial x_n},$$

$$R_{m+3,7}\left(D_x\right)=R_{7,n+3}\left(D_x\right)=R_{m+3,8}\left(D_x\right)=R_{8,n+3}\left(D_x\right)=0,$$

$$R_{7n}\left(D_x\right)=q_{31}\frac{\partial}{\partial x_n},\qquad R_{77}\left(D_x\right)=q_{33}(\Delta),\qquad R_{78}\left(D_x\right)=q_{34}(\Delta),$$

$$R_{8n}\left(D_x\right)=q_{41}\frac{\partial}{\partial x_n},\qquad R_{87}\left(D_x\right)=q_{43}(\Delta),\qquad R_{88}\left(D_x\right)=q_{44}(\Delta),\qquad m,n=1,2,3. \tag{2.59}$$

From Eqs (2.25), (2.55) and (2.58) we obtain

$$\Theta\bigcup = R^{tr}F^{tr}\bigcup \Rightarrow R^{tr}F^{tr} = \Theta,$$

$$\Rightarrow R\left(D_x\right)F\left(D_x\right) = \Theta(\Delta), \tag{2.60}$$

we assume that

$$\lambda_m^2 \neq \lambda_n^2 \neq 0, \qquad m,n = 1,2,3,4,5,6, \qquad m \neq n .$$

Let

$$\mathrm{Y}(x) = \left\|\mathrm{Y}_{rs}(x)\right\|_{8\times 8}, \qquad \mathrm{Y}_{mn}(\boldsymbol{x}) = \sum_{n=1}^{5} r_{1n}\varsigma_n(x),$$

$$\mathrm{Y}_{m+3,n+3}(x) = \sum_{n=4}^{6} r_{2n}\varsigma_n(x), \qquad \mathrm{Y}_{77}(x) = \sum_{n=1}^{3} r_{3n}\varsigma_n(x), \qquad \mathrm{Y}_{88}(x) = \sum_{n=1}^{3} r_{4n}\varsigma_n(x),$$

$$\mathrm{Y}_{vw}(x) = 0, \qquad m = 1,2,3, \qquad v,w = 1,2,..........8, \qquad v \neq w,$$

$$\varsigma_n(x) = -\frac{1}{4\pi|x|}\exp\left(i\lambda_n|x|\right), \qquad n = 1,........,6, \qquad r_{1n} = \prod_{m=1,m\neq 1}^{5}\left(\lambda_n^2 - \lambda_l^2\right)^{-1}, \qquad l = 1,2,.......5,$$

$$r_{2v} = \prod_{m=4,m\neq v}^{5}\left(\lambda_m^2 - \lambda_v^2\right)^{-1}, \qquad v = 4,5,6, \qquad r_{3g} = \prod_{m=1,m\neq g}^{3}\left(\lambda_m^2 - \lambda_g^2\right)^{-1}, \qquad g = 1,2,3,$$

$$r_{3q} = \prod_{m=1,m=q}^{3}\left(\lambda_m^2 - \lambda_q^2\right)^{-1}, \qquad q = 1,2,3 .$$

We will prove the following Lemma:

Lemma: The matrix Y is the fundamental matrix of the operator $\Theta(\Delta)$, i.e.,

$$\Theta(\Delta)\mathrm{Y}(x) = \delta(x)I(x), \tag{2.61}$$

Proof: To prove the above Lemma, it is sufficient to prove that

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\mathrm{Y}_{11}(x) = \delta(x), \tag{2.62}$$

$$\Gamma_1(\Delta)\mathrm{Y}_{77}(x) = \delta(x), \tag{2.63}$$

$$\Gamma_2(\Delta)\left(\Delta + \lambda_6^2\right)\mathrm{Y}_{44}(x) = \delta(x). \tag{2.64}$$

Now we find that

$$r_{11}+r_{12}+r_{13}+r_{14}+r_{15}=r_{12}\left(\lambda_1^2-\lambda_2^2\right)+r_{13}\left(\lambda_1^2-\lambda_3^2\right)+r_{14}\left(\lambda_1^2-\lambda_4^2\right)+r_{15}\left(\lambda_1^2-\lambda_5^2\right)=0\,,$$

$$r_{13}\left(\lambda_1^2-\lambda_3^2\right)\left(\lambda_2^2-\lambda_3^2\right)+r_{14}\left(\lambda_1^2-\lambda_4^2\right)\left(\lambda_2^2-\lambda_4^2\right)+r_{15}\left(\lambda_1^2-\lambda_5^2\right)\left(\lambda_2^2-\lambda_5^2\right)=0\,,$$

$$r_{14}\left(\lambda_1^2-\lambda_2^2\right)\left(\lambda_2^2-\lambda_4^2\right)\left(\lambda_3^2-\lambda_4^2\right)+r_{15}\left(\lambda_1^2-\lambda_5^2\right)\left(\lambda_2^2-\lambda_5^2\right)\left(\lambda_3^2-\lambda_5^2\right)=0\,, \tag{2.65}$$

$$r_{15}\left(\lambda_1^2-\lambda_5^2\right)\left(\lambda_2^2-\lambda_5^2\right)\left(\lambda_3^2-\lambda_5^2\right)\left(\lambda_4^2-\lambda_5^2\right)=1\,,$$

$$\left(\Delta+\lambda_m^2\right)\varsigma_n(x)=\delta(x)+\left(\lambda_m^2-\lambda_n^2\right)\varsigma_n(x), \qquad m,n=1,2,.........5\,.$$

Now consider

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\Upsilon_{11}=\left(\Delta+\lambda_2^2\right)\left(\Delta+\lambda_3^2\right)\left(\Delta+\lambda_4^2\right)\left(\Delta+\lambda_5^2\right)\sum_{n=1}^{5}r_{1n}\left[\delta+\left(\lambda_1^2-\lambda_n^2\right)\varsigma_n\right]=$$

$$=\left(\Delta+\lambda_2^2\right)\left(\Delta+\lambda_3^2\right)\left(\Delta+\lambda_4^2\right)\left(\Delta+\lambda_5^2\right)\sum_{n=2}^{5}r_{1n}\left(\lambda_1^2-\lambda_n^2\right)\varsigma_n=$$

$$=\left(\Delta+\lambda_3^2\right)\left(\Delta+\lambda_4^2\right)\left(\Delta+\lambda_5^2\right)\sum_{n=2}^{5}r_{1n}\left(\lambda_1^2-\lambda_n^2\right)\left[\delta+\left(\lambda_2^2-\lambda_n^2\right)\varsigma_n\right]=$$

$$=\left(\Delta+\lambda_3^2\right)\left(\Delta+\lambda_4^2\right)\left(\Delta+\lambda_5^2\right)\sum_{n=3}^{5}r_{1n}\left(\lambda_1^2-\lambda_n^2\right)\left(\lambda_2^2-\lambda_n^2\right)\varsigma_n=$$

$$=\left(\Delta+\lambda_4^2\right)\left(\Delta+\lambda_5^2\right)\sum_{n=3}^{5}r_{1n}\left(\lambda_1^2-\lambda_n^2\right)\left(\lambda_2^2-\lambda_n^2\right)\left[\delta+\left(\lambda_3^2-\lambda_n^2\right)\varsigma_n\right]=$$

$$=\left(\Delta+\lambda_4^2\right)\left(\Delta+\lambda_5^2\right)\sum_{n=4}^{5}r_{1n}\left(\lambda_1^2-\lambda_n^2\right)\left(\lambda_2^2-\lambda_n^2\right)\left(\lambda_3^2-\lambda_n^2\right)\varsigma_n=$$

$$=\left(\Delta+\lambda_5^2\right)\sum_{n=3}^{5}r_{1n}\left(\lambda_1^2-\lambda_n^2\right)\left(\lambda_2^2-\lambda_n^2\right)\left(\lambda_3^2-\lambda_n^2\right)\left[\delta+\left(\lambda_4^2-\lambda_n^2\right)\varsigma_n\right]=\left(\Delta+\lambda_5^2\right)\varsigma_5=\delta.$$

Equations (2.63) and (2.64) can be proved similarly.
Introducing the matrix

$$G(x)=R(D_x)\Upsilon(x)\,. \tag{2.66}$$

From Eqs (2.60), (2.61) and (2.66), we obtain

$$F(D_x)G(x)=F(D_x)R(D_x)\Upsilon(x)=\Theta(\Delta)\Upsilon(x)=\delta(x)I(x) \tag{2.67}$$

where $G(x)$ is a solution to Eq.(2.20).

Therefore, we proved the following theorem.

Theorem: The matrix $G(x)$ defined by the Eq.(2.66) is the fundamental solution to the system of Eqs (2.12)-(2.15).

## 3. Basic properties of the matrix $G(x)$

Property 1. Each column of the matrix $G(x)$ defined by Eq.(2.66) is the solution to the system of Eqs (2.12)-(2.15) at every point $x \in E^3$ except the origin.

Property 2. The matrix $G(x)$ can be written in the form

$$G = \left\| G_{gh} \right\|_{8\times 8},$$

$$G_{mn}(x) = R_{mn}\left(D_x\right)\Upsilon_{11}(x),$$

$$G_{m,n+3}(x) = R_{m,n+3}\left(D_x\right)\Upsilon_{44}(x),$$

$$G_{m7}(x) = R_{m7}\left(D_x\right)\Upsilon_{77}(x),$$

$$G_{m8}(x) = R_{m8}\left(D_x\right)\Upsilon_{88}(x), \qquad m = 1,2,................8, \qquad n = 1,2,3.$$

If we take $\lambda^*,\mu^*,K^*,\alpha^*,\beta^*,\gamma^*,\xi^*,\varsigma^*,\tau^* \to 0$, we obtain the resulting expressions for micropolar thermoelastic solids with one relaxation time and these results are similar to those obtained by Svandze (2007).

## 4. Conclusion

The fundamental solution $G(x)$ to the system of Eqs (2.12)-(2.15) makes it possible to investigate three-dimensional boundary value problems of the generalized theory of micropolar thermoelastic solids with voids by the potential method Kupradge *et al.* (1979).

## Nomenclature

$C$ – specific heat at constant strain

$j$ – microinertia

$K_1$ – thermal conductivity

$T = \Theta - T_0$ – small temperature increment

$T_0$ – reference temperature of the body choosen such that $\left|\frac{T}{T_0}\right| << 1$

$\boldsymbol{u}$ – displacement vector

$X = \left(x_1, x_2, x_3\right)$ – point of the Euclidean three-dimensional space $E^3$

$|x| = \left(x_1^2, x_2^2, x_3^2\right)^{\frac{1}{2}}$,

$D_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$

$\alpha_t$ – coefficient of linear thermal expension

$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$

$\nabla = \left( \hat{i}\frac{\partial}{\partial x_1} + \hat{j}\frac{\partial}{\partial x_2} + \hat{k}\frac{\partial}{\partial x_3} \right)$

$\Theta$ – absolute temperature of the medium

$\lambda, \mu, K, \alpha, \beta, \gamma, \xi, \varsigma, \lambda^*, \mu^*, K^*, \alpha^*, \beta^*, \gamma^*, \xi^*, \tau^*$ – material constants

$\rho$ – density

$\tau_0$ – relaxation time

$\Phi$ – volume fraction field

$\boldsymbol{\varphi}$ – microrotation vector

$\chi$ – equilibrated inertia

$\omega_1^*$ – characteristic frequency

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